ELSEVIER

Topology and its Applications 68 (1996) 113–131

TOPOLOGY AND ITS APPLICATIONS

Subspaces of connected spaces

Jack R. Porter [a,*], R. Grant Woods [b]

[a] *Department of Mathematics, University of Kansas, Lawrence, KS 66045, USA*
[b] *Department of Mathematics, University of Manitoba, Winnipeg, Man., R3T 2N2, Canada*

Received 8 August 1994; revised 6 June 1995

Abstract

A *connectification* of a topological space X is a connected Hausdorff space that contains X as a dense subspace. Watson and Wilson have noted that a Hausdorff space with a connectification has no nonempty proper clopen H-closed subspaces. Here it is proven that a Hausdorff space in which every nonempty proper clopen set is not feebly compact and the cardinality of the set of clopen sets is at most $2^{\mathfrak{c}}$ is connectifiable. This result is used to show that every metric space with no nonempty proper clopen H-closed subspace is connectifiable, answering a question asked by Watson and Wilson. Also, there is a nonconnectifiable, Hausdorff space of cardinality $\mathfrak{c}$ with no proper H-closed subspace. Using the set-theoretic hypothesis $\mathfrak{p} = \mathfrak{c}$, an example of a nonconnectifiable, normal Hausdorff space of cardinality $\mathfrak{c}$ is constructed which has no nonempty compact open subset. This space is locally compact at all but one point, and if the continuum hypothesis is assumed it is first countable. This space provides a solution to questions asked by Watson and Wilson as well as Mack. The paper concludes by examining when extremally disconnected Tychonoff spaces have Tychonoff connectifications.

Keywords: Connected; Hausdorff extension; H-closed

AMS classification: Primary 54D05; 54D25; 54D35

1. Introduction

A characterization of those spaces which can be densely embedded in a compact Hausdorff space was announced in a 1930 paper [19] by Tychonoff and is familiar to beginning students in topology. A characterization of those spaces which can be densely embedded in some connected Hausdorff space is still unknown. A major step towards solving this problem was taken by Watson and Wilson in a recent paper [20]. In particular,

* Corresponding author. E-mail: porter@kuhub.cc.ukans.edu.

0166-8641/96/$15.00
SSDI 0166-8641(95)00057-7

they completely characterized those countable spaces which can be densely embedded in a connected Hausdorff space.

In this paper we develop more sufficient conditions for a space to be densely embeddable in a connected Hausdorff space and present some limiting counterexamples. The paper is organized as follows. In Section 2 of this paper we present some rather broad sufficient conditions that ensure that a Hausdorff space can be densely embedded in a connected Hausdorff space. Using these, we completely characterize those metric spaces that can be densely embedded in a connected Hausdorff space. In Section 3 we present several examples of "nice" Hausdorff spaces that cannot be densely embedded in a connected Hausdorff space. In particular, using the set-theoretic hypothesis $\mathfrak{p} = \mathfrak{c}$ (discussed in detail in [4]), we produce an example of a normal Hausdorff space of cardinality $\mathfrak{c}$ which has no nonempty compact open subset but which nonetheless cannot be densely embedded in any connected Hausdorff space. This space is locally compact at all but one point, and if the continuum hypothesis is assumed it can be made to be first countable. This space provides a solution to questions in [12] and [20]. In Section 4 we examine which extremally disconnected Tychonoff spaces have connected compactifications. In the final section (Section 5), we answer other questions in [20] by characterizing those spaces that are embeddable as open or closed subspaces of separable connected Hausdorff spaces.

We will assume that *all* hypothesized space are Hausdorff unless specified otherwise. A space X which can be densely embedded in a connected space Y (which is Hausdorff by assumption) is said to be *connectifiable* and Y is called a *connectification* of X. As usual, we let $\mathfrak{c} = 2^{\omega}$, the cardinality of the continuum.

Recall that a space is *H-closed* if it is closed in every space in which it can be embedded and is *feebly compact* if it contains no infinite locally finite family of nonempty open sets. A regular space is H-closed if and only if it is compact (see 4.8(c) of [17]). The set of regular open sets of a space X is denoted by $\mathcal{RO}(X)$ and the set of clopen sets by $\mathcal{B}(X)$ and $\mathcal{B}(X) \setminus \{\emptyset, X\}$ by $\mathcal{C}(X)$. The *weight* of a space X, denoted as wX, is the least cardinal of an open base for X. The *density character* of a space X, denoted as dX, is the least cardinality of a dense subset of X. A family $\mathcal{B}$ of nonempty open subsets of a space is a *π-base* if each nonempty open set of the space contains a member of $\mathcal{B}$; the *π-weight* of a space X, denoted as πwX, is the least cardinality of a π-base for X.

The notation and terminology used in this paper follow [17]. In particular, the collection of open sets of a space X is denoted as $\tau(X)$ and a discrete space of cardinality κ is denoted as $D(\kappa)$. We assume that the reader is familiar with the continuum hypothesis (CH), Martin's Axiom (MA), and the combinatorial principle P($\mathfrak{c}$); these set-theoretic axioms are discussed in Chapter 3 of [17].

The authors appreciate the many helpful suggestions provided by the referee.

Watson and Wilson [20] noted this necessary condition for a space to be connectifiable.

Theorem 1.1. *A connectifiable space contains no proper nonempty H-closed clopen subset.*

2. Sufficient conditions for Hausdorff connectifications to exist

An *open filter* on a space X is a filter on the lattice $(\tau(X), \subseteq)$; an *open ultrafilter* is an open filter not properly contained in any other open filter. An open filter α on X is *free* if $\bigcap\{\mathrm{cl}_X U\colon U \in \alpha\} = \emptyset$. An open filter α on X *meets* a subset T of X if $U \cap T \neq \emptyset$ for each $U \in \alpha$. Two free open filters α and β on X are *Hausdorff separated* if there exists $U \in \alpha$ and $V \in \beta$ with $U \cap V = \emptyset$. An open filter α *converges* to a point p if each open neighborhood of p belongs to α; thus, no open filter converges to more than one point.

Let E be a set of free open filters on a space X and let $Y = X \cup E$. The *simple topology* τ^+ on the set Y is defined as follows:

$$\tau^+ = \{S \subseteq Y\colon \text{ if } \alpha \in S \cap E \text{ then } S \cap X \in \alpha\} \cup \tau(X).$$

One easily verifies that τ^+ is a topology on Y (non-Hausdorff, in general) and that X is a dense open subspace of (Y, τ^+). See [17] for a more detailed discussion of these ideas.

Lemma 2.1. *Let E be a set of free open filters on a space X. Suppose:*

(a) *Each pair of members of E is Hausdorff separated.*

(b) *If $C \in \mathcal{C}(X)$ then there exists $\alpha \in E$ such that both C and $X \setminus C$ meet α.*

Then (Y, τ^+) (as described above) is a connectification of X.

Proof. As noted above, X is a dense subspace of Y. The proof that Y is Hausdorff is straightforward. To show Y is connected, assume there is some $A \in \mathcal{C}(Y)$. Then $A \cap X \in \mathcal{C}(X)$ as X is dense in Y. By (b), there exists $\alpha \in E$ such that $A \cap X$ and $X \setminus A$ meet α. One readily checks that $\alpha \in \mathrm{cl}_Y A \cap \mathrm{cl}_Y(X \setminus A)$, a contradiction. □

Hence to build a connectification of a space X, it suffices to find a set E of free open filters on X that satisfy Lemma 2.1 (a) and (b). We will use this fact several times, beginning with the next result.

Lemma 2.2. *Let X be a space. If there is a one-to-one function $U \mapsto \mathcal{U}_U$ that assigns to each nontrivial clopen subset U of X, a free open ultrafilter that contains U, then X is connectifiable.*

Proof. It is easy to show that $E = \{\mathcal{U}_U \cap \mathcal{U}_{X \setminus U}\colon U \text{ is a nontrivial clopen set}\}$ of free open filters on X that satisfies (a) and (b) of 2.1. Thus, X is connectifiable. □

To apply 2.2, we need a property that leads to the existence of many free open ultrafilters containing a particular nontrivial clopen set.

Lemma 2.3 [17, 7P(2)]. *Let U be a clopen subset of X that is not feebly compact. Then $|\{\mathcal{U}\colon \mathcal{U} \text{ is a free open ultrafilter on } X \text{ and } U \in \mathcal{U}\}| \geqslant 2^{\mathfrak{c}}$.*

Proposition 2.4. *A space X is connectifiable if each nontrivial clopen set is not feebly compact and $|\mathcal{C}(X)| \leqslant 2^{\mathfrak{c}}$.*

Proof. For each nontrivial clopen set U of X, there are at least $2^{\mathfrak{c}}$ free open ultrafilters $\mathcal{U}$ on X with $U \in \mathcal{U}$. Using that $|\mathcal{C}(X)| \leqslant 2^{\mathfrak{c}}$, it is straightforward to establish by induction a one-to-one function $U \mapsto \mathcal{U}_U$ that assigns to each $U \in \mathcal{C}(X)$, a free open ultrafilter $\mathcal{U}_U$ on X that contains U. By 2.2, X is connectifiable. □

The proof of 2.4 is similar to the proof of 3.5 of [20].

Corollary 2.5. *A space X is connectifiable if each nontrivial clopen set is not feebly compact and $dX \leqslant \mathfrak{c}$.*

Proof. Let D be a dense subset of X such that $|D| \leqslant \mathfrak{c}$. The function $U \mapsto U \cap D$ that assigns to each $U \in \mathcal{C}(X)$, a subset of D is one-to-one. Thus, the number of nontrivial clopen subsets of X is at most $2^{\mathfrak{c}}$. By 2.4, X is connectifiable. □

One way of applying 2.5 is to examine those spaces in which feebly compact clopen sets are H-closed. To this end, recall that a space is *almost realcompact* [7] if every free open ultrafilter $\mathcal{U}$ on X has a countable subfamily $\mathcal{G}$ such that $\bigcap\{\operatorname{cl} G\colon G \in \mathcal{G}\}$ is empty.

Corollary 2.6. *An almost realcompact space X with no nontrivial H-closed clopen set is connectifiable if $|\mathcal{C}(X)| \leqslant 2^{\mathfrak{c}}$.*

Proof. Let $U \in \mathcal{C}(X)$. It is immediate from the definition of almost realcompactness that a clopen subset of an almost realcompact space is also almost realcompact; so, U is almost realcompact. As a feebly compact, almost realcompact space is H-closed (see [7, Theorem 2]), it follows that U is not feebly compact. By 2.4, X is connectifiable. □

Comments 2.7. (a) The hypothesis "$|\mathcal{C}(X)| \leqslant 2^{\mathfrak{c}}$" of 2.4 can be replaced with the existence of a family $\mathcal{B}$ of nontrivial clopen sets with the properties that $|\mathcal{B}| \leqslant 2^{\mathfrak{c}}$ and each nontrivial clopen subset of X contains some element of $\mathcal{B}$. In this case, the cardinality of the connectification Y of X will be $|X| \cdot |\mathcal{B}|$. Only a slight modification of the proof of 2.2 is needed. Let $\mathcal{B}' = \{(U,V) \in \mathcal{B}^2\colon U \cap V = \emptyset\}$. Since $|\mathcal{B}'| \leqslant 2^{\mathfrak{c}}$ and since for each $U \in \mathcal{B}$, there are at least $2^{\mathfrak{c}}$ free open ultrafilters $\mathcal{U}$ on X such that $U \in \mathcal{U}$, it is easy to establish, by induction, a one-to-one function $(U,V) \mapsto \mathcal{U}_{(U,V)}$ that assigns to each $(U,V) \in \mathcal{B}'$, a free open ultrafilter $\mathcal{U}_{(U,V)}$ on X such that $U \in \mathcal{U}_{(U,V)}$. It is straightforward to verify that the set $E = \{\mathcal{U}_{(U,V)} \cap \mathcal{U}_{(V,U)}\colon (U,V) \in \mathcal{B}'\}$ of free open filters on X satisfies (a) and (b) of 2.1. By 2.1, X has a connectification Y such that $Y \setminus X = E$. It follows that $|Y| = |X| \cdot |\mathcal{B}|$.

(b) [5] The Sorgenfrey line $\mathbb{S}$ is connectifiable since $\mathbb{S}$ is separable, realcompact (realcompact spaces are almost realcompact by [7, Theorem 10]), and has no nontrivial H-closed clopen subsets.

(c) [20] A countable space is separable and almost realcompact. Recall that a countable H-closed space has a dense set of isolated points (see 5.2 of [16]). So, a countable space without isolated points is connectifiable.

(d) Using 2.5, it is immediate that a metric space with density character at most $\mathfrak{c}$ and with no proper nonempty H-closed clopen sets is connectifiable (since feebly compact subsets of a metric space are compact).

Watson and Wilson [20, Problem 4.8] have asked whether every metric space with no proper nonempty compact open subset is connectifiable, i.e., whether 2.7(d) holds without the "density character at most $\mathfrak{c}$" hypothesis. An affirmative solution is provided after the following lemma.

Lemma 2.8 [20]. *If $\mathcal{F}$ is a locally finite family of nonempty open sets in a space without isolated points, there is a locally finite family $\mathcal{G}(\mathcal{F}) = \{G(F)\colon F \in \mathcal{F}\}$ of pairwise disjoint nonempty open sets such that $G(F) \subset F$ for each $F \in \mathcal{F}$.*

Observe that the local finiteness of $\mathcal{F}$ in 2.8 implies that $|\mathcal{G}(\mathcal{F})| = |\mathcal{F}|$.

Theorem 2.9. *A space X with $\pi w X \geqslant \omega_1$, a σ-locally finite π-base, and no proper nonempty H-closed open subset has a connectification.*

Proof. Now X has a σ-locally finite π-base $\mathcal{C} = \bigcup\{\mathcal{C}_n\colon n \in \omega\}$ where each $\mathcal{C}_n$ is locally finite. As $\pi w X \geqslant \omega_1$, we can assume that $|\mathcal{C}_0| \geqslant \omega_1$. By 2.8 and the observation following 2.8, there is a π-base $\mathcal{B} = \bigcup\{\mathcal{B}_n\colon n \in \omega\}$ such that

(i) each $\mathcal{B}_n$ is a locally finite family of pairwise disjoint open sets and $|\mathcal{B}_n| \geqslant \omega_1$,

(ii) if $n < m$ and $B \in \mathcal{B}_n$, there is some $C \in \mathcal{B}_m$ such that $C \subset B$ ("$\subset$" denotes proper inclusion, so $C \neq B$), and

(iii) if $C \in \mathcal{B}_m$ and $C \cap B \neq \emptyset$ for some $B \in \mathcal{B}_n$ where $n \leqslant m$, then $C \subseteq B$. (Specifically, one constructs the $\mathcal{B}_n$'s as follows: Let $\mathcal{B}_0 = \mathcal{G}(\mathcal{C}_0)$. Let $\mathcal{B}_1^{**} = \mathcal{G}(\mathcal{B}_0 \cup \mathcal{C}_1)$ and $\mathcal{B}_1^* = \{B \cap F\colon B \in \mathcal{B}_0 \text{ and } F \in \mathcal{B}_1^{**}\} \cup \{F \in \mathcal{B}_1^{**}\colon F \cap \bigcup \mathcal{B}_0 = \emptyset\}$; the local finiteness of $\mathcal{B}_1^*$ follows from that of $\mathcal{B}_1^{**}$ and $\mathcal{B}_0$. Since X has no isolated points, we can, if necessary, remove one point from each member of $\mathcal{B}_1^*$ to obtain $\mathcal{B}_1$ and ensure the proper inclusion required in (ii). In general, we proceed inductively and let $\mathcal{B}_{n+1}^{**} = \mathcal{G}(\mathcal{B}_n \cup \mathcal{C}_{n+1})$ and then repeat the above steps to get $\mathcal{B}_{n+1}$.)

By (i) and (ii), note that $\mathcal{B}_n \cap \mathcal{B}_m = \emptyset$ for $n \neq m$. Since X has no isolated points, we can inductively choose points $p_B, q_B \in B$ for each $B \in \mathcal{B}$ such that p_B, q_B, p_C, and q_C are four distinct points whenever $B, C \in \mathcal{B}$ and $B \neq C$.

(iv) If $C \in \mathcal{C}(X)$ and $\pi w C > \omega$, then there is some $n \in \omega$ such that $|\{B \in \mathcal{B}_n\colon B \subseteq C\}| > \omega$. Once n is fixed, it follows by (ii) that $|\{B \in \mathcal{B}_m\colon B \subseteq C\}| > \omega$ for all $m \geqslant n$.

For each $n \in \omega$, let $\mathcal{F}_n = \{U \in \tau(X)\colon \{B \in \mathcal{B}_n\colon p_B \notin U\} \text{ is finite}\}$. Now $\mathcal{F}_n$ is an open filter on X.

(v) $\mathcal{F}_n$ is free since $\mathcal{B}_n$ is a locally finite family.

(vi) If $n < m$, then the two open filters $\mathcal{F}_n$ and $\mathcal{F}_m$ are Hausdorff separated. To verify this, let $B \in \mathcal{B}_n$. Since $\mathcal{B}_m$ is locally finite, there exists $W_B \in \tau(X)$ such that $p_B \in W_B \subseteq B$ and $\{C \in \mathcal{B}_m\colon W_B \cap C \neq \emptyset\}$ is a finite subfamily $\mathcal{F}$ of $\mathcal{B}_m$. As $p_B \notin \{p_C\colon C \in \mathcal{F}\}$ there is an open neighborhood S_B of p_B such that $\{p_C\colon C \in \mathcal{F}\} \cap \operatorname{cl} S_B = \emptyset$. Then $U = \bigcup\{W_B \cap S_B\colon B \in \mathcal{B}_n\} \in \mathcal{F}_n$. By construction and the fact that $\{W_B \cap S_B\colon B \in \mathcal{B}_n\}$ is locally finite, $\operatorname{cl} U \cap \{p_B\colon B \in \mathcal{B}_m\} = \emptyset$. Thus, $X \setminus \operatorname{cl} U \in \mathcal{F}_m$. So $\mathcal{F}_n$ and $\mathcal{F}_m$ are Hausdorff separated.

(vii) For $C \in \mathcal{C}(X)$, observe that for $n \in \omega$, $\mathcal{F}_n$ meets C iff $\{B \in \mathcal{B}_n\colon p_B \in C\}$ is infinite. Let $\mathcal{C} = \{C \in \mathcal{C}(X)\colon$ for all $n \in \omega$, C does not meet $\mathcal{F}_n\}$. Note that for $C \in \mathcal{C}$, $\{B \in \mathcal{B}\colon B \subseteq C\}$ is countable, i.e., $\pi\omega C \leqslant \omega$. Let $\mathcal{M}$ be a maximal pairwise disjoint collection of elements of $\mathcal{C}$. For $n \in \omega$ and $B \in \mathcal{B}_n$, let $\mathcal{A}_B = \{C\colon C$ is clopen and $B \subseteq C \subseteq M$ for some $M \in \mathcal{M}\}$ ($\mathcal{A}_B = \emptyset$ if for each $M \in \mathcal{M}$, $B \nsubseteq M$). Now, $|\mathcal{A}_B| \leqslant \mathfrak{c}$ as B is contained in at most one element of $\mathcal{M}$ and $\pi\omega M \leqslant \omega$ for $M \in \mathcal{M}$. Let $M_{-1} = \emptyset$. Inductively define $\mathcal{M}_n$ by $\mathcal{M}_n = \bigcup\{\mathcal{A}_B\colon B \in \mathcal{B}_n\} \setminus \bigcup\{\mathcal{M}_k\colon k < n\}$. Note that $|\mathcal{M}_n| \leqslant |\mathcal{B}_n| \cdot \mathfrak{c}$ and that $\bigcup\{\mathcal{M}_n\colon n \in \omega\} = \{C\colon C$ is nonempty clopen and $C \subseteq M$ for some $M \in \mathcal{M}\}$ since $\mathcal{B}$ is a π-base for X.

(viii) For $C \in \bigcup\{\mathcal{M}_n\colon n \in \omega\}$, let γ_C be a free open ultrafilter on X such that $C \in \gamma_C$. Now, γ_C and $\mathcal{F}_n$ are Hausdorff separated as $C \in \gamma_C$ and $X \setminus C \in \mathcal{F}_n$.

(ix) Fix $n \in \omega$. Let $\mathrm{UNIF}(\mathcal{B}_n) = \{\mathcal{U}\colon \mathcal{U}$ is a free uniform ultrafilter on the set $\mathcal{B}_n\}$. For $\mathcal{U} \in \mathrm{UNIF}(\mathcal{B}_n)$, let $\mathcal{G}_\mathcal{U} = \{U \in \tau(X)\colon \{B \in \mathcal{B}_n\colon q_B \in U\} \in \mathcal{U}\}$. Using the same proof as (v), one can show that the open filter $\mathcal{G}_\mathcal{U}$ on X is free. Also, using a proof similar to the proof of (vi), one can show that for $n, m \in \omega$ and $\mathcal{U} \in \mathrm{UNIF}(\mathcal{B}_n), \mathcal{V} \in \mathrm{UNIF}(\mathcal{B}_m)$ the following are true:

(a) $\mathcal{G}_\mathcal{U}$ and $\mathcal{F}_m$ are Hausdorff separated (in the case of $m = n$, the proof uses that $p_B \neq q_B$ for $B \in \mathcal{B}_n$),

(b) for $n \neq m$, $\mathcal{G}_\mathcal{U}$ and $\mathcal{G}_\mathcal{V}$ are Hausdorff separated (the proof is similar to the proof that $\mathcal{F}_n$ and $\mathcal{F}_m$ are Hausdorff separated and uses that $\mathcal{B}_n \cap \mathcal{B}_m = \emptyset$ and that $q_B \neq q_C$ for $B, C \in \mathcal{B}$ and $B \neq C$), and

(c) for $n = m$ and $\mathcal{U} \neq \mathcal{V}$, $\mathcal{G}_\mathcal{U}$ and $\mathcal{G}_\mathcal{V}$ are Hausdorff separated.

(x) Let $M \in \mathcal{M}$, $C \in \mathcal{C}(M)$, and $n \in \omega$. By (vii), $|\{B \in \mathcal{B}_n\colon B \cap C \neq \emptyset\}| \leqslant \omega$. So, if $\mathcal{U} \in \mathrm{UNIF}(\mathcal{B}_n)$, then $\mathcal{B}_n \setminus \{B \in \mathcal{B}_n\colon B \cap C \neq \emptyset\} \in \mathcal{U}$. Thus, $U = \bigcup\{B \in \mathcal{B}_n\colon B \cap C = \emptyset\} \in \mathcal{G}_\mathcal{U}$, $U \cap C = \emptyset$, and $X \setminus C \in \mathcal{G}_\mathcal{U}$. In particular, γ_C and $\mathcal{G}_\mathcal{U}$ are Hausdorff separated. Since γ_C and $\mathcal{G}_\mathcal{U}$ are free, so is $\gamma_C \cap \mathcal{G}_\mathcal{U}$. Also, note that if D is nonempty and clopen and $\pi\omega D \leqslant \omega$, then $X \setminus D \in \mathcal{G}_\mathcal{U}$.

Recall that

$$|\mathrm{UNIF}(\mathcal{B}_n)| = 2^{2^{|\mathcal{B}_n|}}$$

by 7.8 in [2]; so, by (vii), $|\mathrm{UNIF}(\mathcal{B}_n)| \geqslant |\mathcal{M}_n|$. Consider any one-to-one function: $\mathcal{M}_n \to \mathrm{UNIF}(\mathcal{B}_n)$; let $\mathcal{U}_C$ denote the image of C in $\mathcal{M}_n$. Denote $\mathcal{G}_{\mathcal{U}_C}$ by $\mathcal{G}_C$.

(xi) Let $C, D \in \bigcup\{\mathcal{M}_n\colon n \in \omega\}$ such that $\gamma_C \neq \gamma_D$. By (x), γ_C and $\mathcal{G}_D$ are Hausdorff separated, and by (ix)(b,c), $\mathcal{G}_C$ and $\mathcal{G}_D$ are Hausdorff separated. So, there are $U \in \gamma_C$, $V \in \gamma_D$, $W \in \mathcal{G}_C$, and $R \in \mathcal{G}_D$ such that U, V, W, and R are pairwise disjoint. Since $(U \cup W) \cap (V \cup R) = \emptyset$, it follows that $\gamma_C \cap \mathcal{G}_C$ and $\gamma_D \cap \mathcal{G}_D$ are Hausdorff separated.

For $C, D \in \bigcup\{\mathcal{M}_n\colon n \in \omega\}$, it may happen that $\gamma_C = \gamma_D$ (in this case, there is a unique $M \in \bigcup\{\mathcal{M}_n\colon n \in \omega\}$ such that $C, D \subseteq M$). For $C \in \bigcup\{\mathcal{M}_n\colon n \in \omega\}$, we choose one element $H(C) \in \{D \in \bigcup\{\mathcal{M}_n\colon n \in \omega\}\colon \gamma_C = \gamma_D\}$. In particular, for $C, D \in \bigcup\{\mathcal{M}_n\colon n \in \omega\}$, $\gamma_C = \gamma_D$ if and only if $H(C) = H(D)$.

Finally, we show that $E = \{\mathcal{F}_n\colon n \in \omega\} \cup \{\gamma_{H(C)} \cap \mathcal{G}_{H(C)}\colon C \in \bigcup\{\mathcal{M}_n\colon n \in \omega\}\}$ satisfies 2.1 (a) and (b). For $n \in \omega$ and $C \in \bigcup\{\mathcal{M}_n\colon n \in \omega\}$, by (ix)(a) $\mathcal{F}_n$ and $\mathcal{G}_{H(C)}$ are Hausdorff separated and by (viii), $\mathcal{F}_n$ and $\gamma_{H(C)}$ are Hausdorff separated. It follows that $\mathcal{F}_n$ and $\gamma_{H(C)} \cap \mathcal{G}_{H(C)}$ are Hausdorff separated. If $n, m \in \omega$ and $n \neq m$, $\mathcal{F}_n$ and $\mathcal{F}_m$ are Hausdorff separated by (vi). If $C, D \in \bigcup\{\mathcal{M}_n\colon n \in \omega\}$ and $H(C) \neq H(D)$, $\gamma_{H(C)} \cap \mathcal{G}_{H(C)}$ and $\gamma_{H(D)} \cap \mathcal{G}_{H(D)}$ are Hausdorff separated by (xi). Thus, E satisfies 2.1(a).

To show 2.1(b) is satisfied, let $C \in \mathcal{C}(X)$. If there are $m, n \in \omega$ such that $\mathcal{F}_n$ meets C and $\mathcal{F}_m$ meets $X \setminus C$, then by (ii), $\mathcal{F}_{\max\{n,m\}}$ meets both C and $X \setminus C$. So, we can assume that C (or $X \setminus C$) does not meet $\mathcal{F}_n$ for all $n \in \omega$. Suppose C does not meet $\mathcal{F}_n$ for all $n \in \omega$ (the same proof works for $X \setminus C$). In particular, C meets some $M \in \mathcal{M}$. By (viii), $C \in \gamma_{H(C \cap M)}$ and by (x), $X \setminus C \in \mathcal{U}_{H(C \cap M)}$. It follows that C and $X \setminus C$ meet $\gamma_{H(C \cap M)} \cap \mathcal{U}_{H(C \cap M)}$. This completes the proof that E satisfies 2.1(b). By 2.1, X has a connectification. □

Corollary 2.10. (1) *Every metric space without compact open subsets has a connectification.*

(2) *Every perfect irreducible preimage of a nowhere locally compact metric space has a connectification.*

Proof. (1) A metric space has a σ-locally finite base so our result follows from 2.7(d) and 2.9.

(2) Let X be a nowhere locally compact metric space and let $f : Y \to X$ be a perfect irreducible continuous surjection. Then X has a σ-locally finite base $\mathcal{B}$. Consequently $\{f^{\leftarrow}[B]\colon B \in \mathcal{B}\}$ is a π-base for Y (see 6B(4) of [17]) and it is easily verified to be σ-locally finite in Y (because $\mathcal{B}$ is in X).

If A is clopen in Y, then $f[A]$ is a regular closed subset of X (see 6.5(d)(5) of [17]); as X is nowhere locally compact, $f[A]$ is not compact. As X is metric, $f[A]$ is not feebly compact. Since $f|A : A \to f[A]$ is a perfect irreducible surjection, it follows from 6B(3) of [17] that A is not feebly compact. It now follows from 2.5 and 2.9 that Y has a connectification. □

3. Nonconnectifiable examples

Watson and Wilson [20, Problem 4.5] have asked if there is a nonconnectifiable Tychonoff space of cardinality at most $\mathfrak{c}$ without any proper nonempty compact open subsets. We show there is a nonconnectifiable Hausdorff space of cardinality $\mathfrak{c}$ with no proper nonempty H-closed clopen subsets and, assuming $\mathrm{P}(\mathfrak{c})$, there is a nonconnectifiable, normal Hausdorff space of cardinality $\mathfrak{c}$ with no proper nonempty H-closed clopen subset. If CH is assumed, there is a space with these properties that is also first countable. First, some machinery developed in [20] is needed. Recall that a space is *almost H-closed* if for every pair of disjoint open sets, the closure of one of them is H-closed. As noted in 7P in [17], a space is almost H-closed iff the remainder of every extension is a singleton or the empty set.

The following is a slight generalization of [20, 4.4].

Theorem 3.1. *Let $\{X_\alpha\colon\ \alpha \in I\}$ be a set of almost H-closed spaces, and let*

$$Y = \left(\bigoplus\{X_\alpha\colon\ \alpha \in I\}\right) \cup \{\infty\}.$$

A set $U \subseteq Y$ is defined to be open if for each $\alpha \in I$, $X_\alpha \cap U$ is open in X_α and $\infty \in U$ implies there is a finite subset F_U of I such that $\bigcup\{X_\alpha\colon\ \alpha \in I \setminus F_U\} \subseteq U$. Then Y is a Hausdorff space and Y is not connectifiable. Also, if each X_α has no nonempty H-closed open subset, then Y has no proper nonempty H-closed clopen subset.

It is easy to verify that the space Y described in 3.1 is normal (respectively, Tychonoff) if and only if each X_α is normal (respectively, Tychonoff). Furthermore, if $I = \omega$ and X_α is first countable for each $\alpha \in \omega$, then Y is first countable.

Observe that an almost H-closed space is connected iff it has no proper nonempty H-closed clopen subset. Thus by 3.1, if κ is a cardinal and X is a connected, almost H-closed space, we can generate a nonconnectifiable space Y of cardinality $\max\{\kappa, |X|\}$ by letting $I = \kappa$ and letting X_α be a copy of X for each $\alpha \in I$. Furthermore, if X has no H-closed open subspace, then Y will have no H-closed clopen subset. This technique will now be used to produce examples witnessing that neither of the cardinality hypotheses of 2.9 can be dropped.

For a Tychonoff space X, recall that a point $p \in \beta X \setminus X$ is called a *remote point* of βX if p is not in the βX-closure of any closed nowhere dense subset of X; see [3] for a discussion of remote points. It is easy to see that if X is normal and p is a remote point of βX, then $\beta X \setminus \{p\}$ is almost H-closed but not H-closed.

Examples 3.2. (1) [20] In 3.1 let $I = \omega$ and let each X_n be $\beta H \setminus \{p\}$, where $H = [0, 1)$ and p is a remote point of βH. As $\beta H \setminus \{p\}$ has no H-closed open subspace (being connected, Tychonoff, and noncompact), by the remarks above, the associated space Y described in 3.1 will have no connectification, and no clopen subset that is H-closed. Furthermore, as the free union of a countable number of copies of H form a dense subspace of Y, it is clear that $\pi\omega(Y) = \omega$. Thus Y has a σ-locally finite π-base (each of whose locally finite parts contain precisely one set). This shows that the condition that $\pi\omega X \geqslant \omega_1$ cannot be dropped in 2.9.

(2) In 3.1 let I be the cardinal $\mathfrak{c}$ (viewed as an ordinal) and let X_α be $\beta H \setminus \{p\}$ (as in (1)) for each $\alpha \in I$. Then the associated space Y described in 3.1 will have no connectification and no clopen set that is H-closed. Furthermore, it is clear that $\pi\omega Y = \mathfrak{c}$. One can verify directly that Y does not have a σ-locally finite π-base; this example

illustrates that the existence of such a π-base cannot be dropped from the hypotheses of 2.9. $\square$

Remark 3.3. Note that 2.2 of [20] essentially says that if a space with a σ-discrete π-base has no H-closed regular closed subsets, then it is connectifiable. Theorem 2.9 above says that a space with a σ-locally finite π-base, uncountable π-weight, and no H-closed clopen sets is connectifiable. It is tempting to conjecture that their obvious common generalization is true, viz., if a space with a σ-locally finite π-base has no H-closed clopen sets, then it is connectifiable. However, Example 3.2(1) show that this fails.

Observe that both examples in 3.2 have cardinality $2^{\mathfrak{c}}$, since $|\beta H \setminus \{p\}| = 2^{\mathfrak{c}}$. Before constructing a connected, almost H-closed space X of size $\mathfrak{c}$, one preliminary result is needed. For an H-closed extension Y of a space X, recall that there is a continuous function $f_Y : \kappa Y \to Y$ such that $f_Y(x) = x$ for each $x \in X$. (Here, κX denotes the Katětov H-closed extension of X; see 4.8 of [17].) Also, let σX denote the Fomin H-closed extension of X; see, 7.2 of [17] for a description of its properties. Since κX and σX have the same underlying set of points, $P(Y) = \{f_Y^{\leftarrow}(p)\colon p \in Y \setminus X\}$ is a partition of $\sigma X \setminus X$ into compact subsets by 7.4(a)(1) of [17].

Lemma 3.4. *Let X be a zero-dimensional, compact space and $x \in X$. There is a partition $\{C_\alpha\colon \alpha \leqslant wX\}$ of X into compact subsets such that $C_{wX} = \{x\}$.*

Proof. Let $\lambda = wX$. Using standard evaluation mapping techniques, we can consider X as a subspace of $\mathbf{2}^\lambda$, where $\mathbf{2} = \{0, 1\}$ with the discrete topology. Also, since $\mathbf{2}^\lambda$ is homogeneous, we can assume that $x = \mathbf{0}$ where $\mathbf{0}$ is the constant function. For $\alpha < \lambda$, let $\pi_\alpha : \mathbf{2}^{\mathfrak{c}} \to \mathbf{2}$ be the αth projection and $C_\alpha = \pi_\alpha^{\leftarrow}\{1\} \cap \bigcap\{\pi_\beta^{\leftarrow}\{0\}\colon \beta < \alpha\}$ (in particular, $C_0 = \pi_0^{\leftarrow}\{1\}$) and $C_\lambda = \{\mathbf{0}\}$. Now, $\{X \cap C_\alpha\colon \alpha \leqslant \lambda\}$ is the desired partition of X into compact subspaces. $\square$

Theorem 3.5. *There is a connected, almost H-closed space of cardinality $\mathfrak{c}$ which is not H-closed.*

Proof. Let $\mathbb{R}$ denote the space of all real numbers with the usual topology. Now, by 7.2(c) of [17], $\sigma\mathbb{R} \setminus \mathbb{R}$ is homeomorphic to the zero-dimensional, compact Hausdorff subspace $\beta E\mathbb{R} \setminus E\mathbb{R}$ of $\beta E\mathbb{R}$. Here $E\mathbb{R}$ denotes the absolute of $\mathbb{R}$; see Chapter 6 of [17] for a detailed description of it. Also as $\beta E\mathbb{R}$ is separable, $w(\beta E\mathbb{R}) \leqslant \mathfrak{c}$. Let $y \in \sigma\mathbb{R} \setminus \mathbb{R}$; by 3.4, there is a partition $\{C_\alpha\colon \alpha \leqslant \mathfrak{c}\}$ of $\sigma\mathbb{R} \setminus \mathbb{R}$ into compact subsets such that $C_{\mathfrak{c}} = \{y\}$. By 7.4(a)(2) of [17], there is a largest H-closed extension Y of $\mathbb{R}$ such that $P(Y) = \{C_\alpha\colon \alpha \leqslant \mathfrak{c}\}$. Let $X = Y \setminus \{y\}$. Clearly X is connected and has size $\mathfrak{c}$. Let U and V be disjoint nonempty open subsets of X. Now $f_Y^{\leftarrow}[U]$ and $f_Y^{\leftarrow}[V]$ are disjoint open sets in $\kappa\mathbb{R}$. Since y is an open ultrafilter on $\mathbb{R}$, either $y \notin \mathrm{cl}_{\kappa\mathbb{R}} f_Y^{\leftarrow}[U]$ or $y \notin \mathrm{cl}_{\kappa\mathbb{R}} f_Y^{\leftarrow}[V]$. Suppose $y \notin \mathrm{cl}_{\kappa\mathbb{R}} f_Y^{\leftarrow}[U]$. Then $f[\mathrm{cl}_{\kappa\mathbb{R}} f_Y^{\leftarrow}[U]] = \mathrm{cl}_Y U$ is H-closed and $y \notin \mathrm{cl}_Y U$ since $C_{\mathfrak{c}} = \{y\}$. So, $\mathrm{cl}_X U = \mathrm{cl}_Y U$ is H-closed. Thus, X is almost H-closed. $\square$

Note that 3.1 and 3.5 together provide us with an example of a nonconnectifiable Hausdorff space of cardinality $\mathfrak{c}$ with no proper H-closed clopen subset.

Now we will construct (using the set-theoretic hypothesis $\mathrm{P}(\mathfrak{c})$) a normal Hausdorff space of cardinality $\mathfrak{c}$ with no compact open subsets and no connectification. If CH is assumed, this space will also be first countable. Assuming $\mathrm{P}(\mathfrak{c})$, this example will answer Problem 4.5 of [20]; assuming CH, it simultaneously answers Problems 4.5 and 4.7 of [20].

By 3.1 and the remarks following it, it will suffice to build an almost H-closed, noncompact connected normal Hausdorff space of cardinality $\mathfrak{c}$ that will be first countable when CH is assumed. This we proceed to do.

Lemma 3.6 [$\mathrm{P}(\mathfrak{c})$]. *Let $H = [0, 1)$. Then $H^* = \beta H \setminus H$ contains a "remote $P_{\mathfrak{c}}$-point", i.e., a point p such that $p \notin \mathrm{cl}_{\beta H} D$ if D is a closed nowhere dense subset of H, and if $\kappa < \mathfrak{c}$ and $(U_\alpha)_{\alpha<\kappa}$ is a collection of κ open subsets of H^* with $p \in \bigcap\{U_\alpha\colon \alpha < \kappa\}$, then $p \in \mathrm{int}_{H^*} \bigcap\{U_\alpha\colon \alpha < \kappa\}$.*

Proof. The proof of this lemma is essentially the same as that outlined in Problem 6AD of [17] to show that if MA holds then $\beta X \setminus X$ contains a dense set of P-points if X is a locally compact σ-compact nonpseudocompact space with a countable π-base. □

We remark that the existence of a remote P-point of $\beta H \setminus H$ has been proved by Plank (see [14]), assuming that the continuum hypothesis holds, by a method similar to, but simpler than that outlined above.

Theorem 3.7 [$\mathrm{P}(\mathfrak{c})$]. *There is a connected, almost H-closed, normal space of cardinality $\mathfrak{c}$ which is not H-closed. Under the assumption of CH, this space is also first countable.*

Proof. Let $H = [0, 1)$ and $H^* = \beta H \setminus H$. By 3.6, there is a $p \in H^*$ such that p is a $\mathrm{P}_{\mathfrak{c}}$-point of H^* and p is a remote point of βH.

Using the fact that p is a $\mathrm{P}_{\mathfrak{c}}$-point of H^*, inductively construct a collection $(A_\alpha)_{\alpha<\mathfrak{c}}$, of zero-sets of H^* such that

(i) $A_0 \neq H^*$,

(ii) $\alpha < \delta$ implies $A_\delta \subseteq \mathrm{int}_{H^*} A_\alpha$, and

(iii) $\{\mathrm{int}_{H^*} A_\alpha\colon \alpha < \mathfrak{c}\}$ is a neighborhood base at p in H^*. (In particular,

$$\bigcap\{A_\alpha\colon \alpha < \mathfrak{c}\} = \bigcap\{\mathrm{int}_{H^*} A_\alpha\colon \alpha < \mathfrak{c}\} = \{p\}.)$$

Using that H^* is normal and $A_{\alpha+1}$ is a zero-set of H^*, there is a continuous function $f_\alpha : H^* \to [0, 1]$ for each $\alpha < \mathfrak{c}$ for which

(a) $f_\alpha[H^* \setminus \mathrm{int}\, A_\alpha] = \{0\}$, and

(b) $A_{\alpha+1} = f_\alpha^{\leftarrow}(1)$.

For $x \in H^* \setminus \{p\}$, let

$$\alpha(x) = \min\{\alpha < \mathfrak{c}\colon x \notin A_{\alpha+1}\};$$

by (ii) and (iii) $\alpha(x)$ is well-defined. The following two facts are easy to verify for $x \in H^* \setminus \{p\}$:

(c) $\alpha < \alpha(x)$ iff $x \in A_{\alpha+1}$ iff $f_\alpha(x) = 1$.

(d) If $f_\alpha(x) > 0$, then $x \in A_\alpha$ implying $\alpha(x) \geqslant \alpha$.

Let L be the "$\mathfrak{c}$-long line", i.e., $L = [0, \mathfrak{c}) \times [0, 1)$ with the lexicographic order. Then $\beta L = L \cup \{\infty\}$ (see, 16H of [9]), and βL is, in fact, an ordered space with ∞ being the "largest element".

Define $F : H^* \to \beta L$ by $F(x) = (\alpha(x), f_{\alpha(x)}(x))$ for $x \in H^* \setminus \{p\}$ and $F(p) = \infty$. To verify that F is well-defined it suffices to check that for each $x \in H^* \setminus \{p\}$, $f_{\alpha(x)}(x) \neq 1$ – this follows from (c). Therefore, $f(z) \in \beta L$ for all $z \in H^*$.

Claim. *F is continuous and onto.*

Proof. As βL is an ordered space, it suffices to show that inverse images of open right rays and open left rays are open. Let $(\alpha, r) \in L \setminus \{(0, 0)\}$ and $x \in H^* \setminus \{p\}$. Using (c) and (d) above it can be proven in a straightforward fashion that if $(\alpha, r) \in \beta L$, then

$$F^{\leftarrow}\big[((\alpha, r), \infty]\big] = f_\alpha^{\leftarrow}\big[(r, 1]\big] \quad \text{and} \quad F^{\leftarrow}\big[[(0,0), (\alpha, r))\big] = f_\alpha^{\leftarrow}\big[[0, r)\big].$$

This completes the proof that F is continuous. Since H^* is connected (see 6.10 in [9]), $F[H^*]$ is a connected subspace of the ordered space βL containing the first element $(0, 0)$ (as $A_0 \neq H^*$) and the last element ∞. Thus, $F[H^*] = \beta L$.

By Magill's theorem [13, Theorem 2.1], there is a compactification αH of H of the form $H \cup \beta L$ where the Čech map $g : \beta H \to \alpha H$ is given by $g|H^* = F$ (and $g|H$ is the identity on H). Let $X = \alpha H \setminus \{\infty\}$. Clearly, X is locally compact and connected, and $|X| = |L| = \mathfrak{c}$. It is easy to show that the continuous image of an almost H-closed set is almost H-closed; since $g^{\leftarrow}(\infty) = \{p\}$, it follows that X is almost H-closed.

Claim. *If A is a closed subset of X such that $A \cap L$ is compact, then A is compact.*

Proof. Since X is locally compact, there is an open set U in X such that $\mathrm{cl}_X U$ is compact and $A \cap L \subseteq U$. Now $A \setminus U \subseteq A \cap H$ and $A \setminus U$ is closed in X. It suffices to show that $A \setminus U$ is compact. Assume not. Then $A \setminus U$ contains an unbounded closed discrete subset S of H. There are two disjoint open sets V and W of H such that $\mathrm{cl}_H V \cap \mathrm{cl}_H W \supseteq S$. Since X is almost H-closed, $\mathrm{cl}_X V$ or $\mathrm{cl}_X W$ is compact. In particular, S is compact, a contradiction as S is an unbounded subset of H.

Claim. *X is normal.*

Proof. Since L is normal and almost compact, at least one of two disjoint closed subsets of L is compact. Thus, at least one of two disjoint closed subsets of X will satisfy the hypothesis of the preceding claim, and hence be compact. It follows that X is normal.

Therefore, X is a normal, almost compact (in fact, almost H-closed), Hausdorff space of cardinality $\mathfrak{c}$. If the continuum hypothesis holds, then $L = [0, \omega_1) \times [0, 1)$ and it is well known that L is first countable. As H is first countable and σ-compact, it follows that each point of X is a G_δ-point of X. Since G_δ-points of locally compact spaces are points of first countability, it follows that X is first countable. □

Theorem 3.8. *If* P($\mathfrak{c}$) *is assumed, then there is a normal Hausdorff space X of cardinality $\mathfrak{c}$ with no compact open sets that cannot be embedded densely in a connected Hausdorff space; furthermore, each point of X is of character $< \mathfrak{c}$. Hence if the continuum hypothesis is assumed, then this space is also first countable.*

Proof. This follows from 3.1 and the remarks following it, together with 3.7 and the remarks preceding 3.6. □

Remark 3.9. (1) Let L^+ and L^- be two copies of the $\mathfrak{c}$-long line L described in 3.7. The double edged $\mathfrak{c}$-long line M is $L^+ \oplus L^-$ with the 0 points of L^+ and L^- identified. Now, βM is the two point compactification of M; the two points are denoted as $\pm\infty$. The construction described in 3.7 can be modified so that $\alpha H \setminus H = \beta M$. Under the assumption of CH, $\alpha H \setminus \{\pm\infty\}$ is normal and pseudocompact, and so, is countably compact.

In 4.2 of [12], Mack produces (by techniques quite different from ours) a compactification γH of H such that $\gamma H \setminus H = \beta M'$ where M' is the double edge ω_1-long line, and so that $Y = \beta M' \setminus \{-\infty, \infty\}$ is a locally compact, normal, first countable space that is locally embeddable in $\mathbb{R}^2$. He then asks if such a Y can be produced, that is, in addition, C^*-embedded in γH (and hence countably compact). Clearly under CH the space $\alpha H \setminus \{-\infty, \infty\}$ produced above is such a space. (The proof that $\alpha H \setminus \{-\infty, \infty\}$ is locally embeddable in $\mathbb{R}^2$ is identical to that produced by Mack for his Y.)

(2) The referee has noted that a first countable example of the space described in 3.8 can exist under any cardinal arithmetic. Starting from any model one can obtain by a ccc forcing of size $\mathfrak{c}$ a remote P-point of character ω_1 in H^*; this is all that is needed to get the first countable example. (See [10].)

4. Tychonoff connectifications of Tychonoff extremally disconnected spaces

Recall (see Chapter 6 of [17], for example) that a space is *extremally disconnected* if its open sets have open closures. In this section we consider when various classes of extremally disconnected Tychonoff spaces have Tychonoff connectifications (equivalently, connected compactifications). As extremally disconnected Tychonoff spaces are zero-dimensional (see 6.4 of [17]), it is clear that an extremally disconnected Tychonoff space that is locally compact at some point will have a nonempty compact open subset and hence not be connectifiable by 1.1. Consequently we can confine our attention to nowhere locally compact extremally disconnected Tychonoff spaces. As extremally disconnected spaces are "as disconnected as possible" in some sense, one's intuition might be that they will have no connected compactifications. In fact, this is the case for one of the two classes that we will consider, but not the other.

We begin by reviewing some known facts. Associated with each regular space X there is an extremally disconnected space EX and a perfect irreducible continuous surjection $k_X : EX \to X$; this pair is essentially unique. The pair (EX, k_X) is called the *absolute* of X. If X and Y are two spaces for which EX and EY are homeomorphic, we say that X and Y are *coabsolute*. The reader is referred to Chapter 6 of [17] for background on absolutes. We will make use of the following well-known properties of absolutes, irreducible maps, and compactifications, which we collect into one lemma for administrative convenience.

Lemma 4.1. (1) *Let αX be a compactification of a Tychonoff space X. Then $\alpha X \setminus X$ is dense in αX if and only if X is nowhere locally compact* (*see* [11, p. 109]).

(2) *If there is a perfect irreducible continuous surjection from Y onto Z, then Y and Z are coabsolute.*

(3) *A Tychonoff space S is nowhere locally compact* (*respectively pseudocompact*) *if and only if ES is nowhere locally compact* (*respectively pseudocompact*). (*See* 2.5 *of* [21] *for "pseudocompact".*)

(4) *If $f : Y \to Z$ is a perfect irreducible continuous surjection, and if S is a dense subset of Z, then $f^{\leftarrow}[S]$ is dense in Y and $f|f^{\leftarrow}[S] : f^{\leftarrow}[S] \to S$ is a perfect continuous irreducible surjection.*

(5) *If X is a dense subspace of the extremally disconnected space T, then X is extremally disconnected and C^*-embedded in T.*

(6) *If X is dense in T, then $k_X^{\leftarrow}[X]$ is dense in ET; hence by* (4) *and* (5), *it is an extremally disconnected dense C^*-embedded subspace of ET, and is homeomorphic to EX. In particular, $\beta(EX) = E(\beta X)$.*

To prove our first result, we need a lemma of van Douwen (see 12.3 of [3]).

Lemma 4.2. *Let αX be a compactification of the nowhere locally compact space X. If $\alpha X \setminus X$ is extremally disconnected, then $\alpha X = \beta X$.*

In 13.2 of [3] van Douwen provides an example of a nowhere locally compact Tychonoff space with no connected compactification. In the following we use some of the ideas employed in his construction to produce a significant generalization of his construction.

Lemma 4.3. *Let X be a nowhere locally compact, extremally disconnected Tychonoff space such that $\beta X \setminus X$ is not pseudocompact. Then X has no connected compactification.*

Proof. Let αX be a compactification of X and denote $\alpha X \setminus X$ by Z. Let $f : \beta X \to \alpha X$ be the Stone extension of the identity map on X. Clearly, f is irreducible, so by 4.1 (1) and (4) $f|(\beta X \setminus X)$ is a perfect irreducible surjection from $\beta X \setminus X$ onto Z. Hence by

4.1(2) it follows that $E(\beta X \setminus X) = EZ$. Thus as $\beta X \setminus X$ is not pseudocompact by hypothesis, by 4.1(3) Z is not pseudocompact either.

By 4.1(1) Z is dense in αX and so αX is a compactification δZ of Z. As $X = \delta Z \setminus Z$ is extremally disconnected, by 4.2 $\delta Z = \beta Z$. Now suppose that αX is connected. By 6L(1) of [9], Z is connected. As Z fails to be pseudocompact, there exists $f \in C(Z)$ and a subset $\{z_n\colon n \in \omega\}$ of Z such that $1 < f(z_n) < f(z_{n+1}) - 1$ for each $n \in \omega$. Let $V_n = \operatorname{int}_Z \operatorname{cl}_Z f^{\leftarrow}[(f(z_n) - 1/4, f(z_n) + 1/4)]$, and let $V = \bigcup\{V_n\colon n \in \omega\}$. As Z is connected, $\operatorname{bd}_Z V_n = \operatorname{cl}_Z V_n \setminus V_n$ is nonempty for each $n \in \omega$. As $\{V_n\colon n \in \omega\}$ is a pairwise completely separated (in Z) family of regular open subsets of Z, it is straightforward to verify that V is a regular open subset of Z and that $\operatorname{bd}_Z V = \bigcup\{\operatorname{bd}_Z V_n\colon n \in \omega\}$. Evidently $f | \operatorname{bd}_Z V$ is unbounded and so $\operatorname{bd}_Z V$ is not compact. Consequently there exists $p \in \operatorname{cl}_{\beta Z}(\operatorname{bd}_Z V) \setminus Z = X \cap \operatorname{cl}_{\beta Z}(\operatorname{bd}_Z V)$. Let $p \in W \in \tau(\beta Z)$. Then $W \cap \operatorname{bd}_Z V \neq \emptyset$ so $\emptyset \neq W \cap V \subseteq W \cap \operatorname{int}_{\beta Z} \operatorname{cl}_{\beta Z} V$. It follows that $W \cap X \cap \operatorname{int}_{\beta Z} \operatorname{cl}_{\beta Z} V \neq \emptyset$ and that $p \in \operatorname{cl}_X(X \cap \operatorname{int}_{\beta Z} \operatorname{cl}_{\beta Z} V)$. Similarly, $p \in \operatorname{cl}_X(X \cap \operatorname{int}_{\beta Z} \operatorname{cl}_{\beta Z}(Z \setminus \operatorname{cl}_Z V))$. Hence X has disjoint open subsets whose closures are not disjoint, contradicting the extremal disconnectedness of X. Consequently, αX is not connected. □

We can use Theorem 4.3 to produce a class of nowhere locally compact Tychonoff spaces whose absolutes do not have connected compactifications.

Corollary 4.4. *Let X be a nowhere locally compact Tychonoff space and have a point of first countability. Then its absolute EX does not have a connected compactification.*

Proof. Let $p \in X$ and let $\{V_n\colon n \in \omega\}$ be a countable open neighborhood base at p in X. Clearly, $\{p\} = \bigcap\{\operatorname{int}_{\beta X} \operatorname{cl}_{\beta X} V_n\colon n \in \omega\}$.

As noted in 4.1(6), $k_{\beta X}^{\leftarrow}[X] = EX$ (up to homeomorphism). Consider the set

$$G = \bigcap\{\operatorname{int}_{\beta X} \operatorname{cl}_{\beta X} V_n\colon n \in \omega\}.$$

Then $k_{\beta X}^{\leftarrow}[G] = k_{\beta X}^{\leftarrow}(p)$ is a G_δ-set of βEX that is disjoint from $\beta EX \setminus EX = k_{\beta X}^{\leftarrow}[\beta X \setminus X]$. But as X is nowhere locally compact, by 4.1(1), (2), (3) and (4) $\beta(\beta EX \setminus EX) = \beta EX$, so the Stone–Čech remainder of $\beta EX \setminus EX$ contains a G_δ-set of βEX that misses $\beta EX \setminus EX$. Thus, $\beta EX \setminus EX$ is not pseudocompact (see 5F(4) of [17], for example). Hence by 4.3, it follows that EX has no connected compactification. □

So, by 4.4, $E\mathbb{Q}$ does not have a connected compactification, however, by 2.9(b), $E\mathbb{Q}$ does have a Hausdorff connectification. In the preceding result we cannot replace "has a point of first countability" by "has a G_δ-point"; see the example following Theorem 4.7 below.

We now consider a class of extremally disconnected spaces that do have connected compactifications. In 13.1 of [3] van Douwen exhibits an extremally disconnected space that has a connected compactification. In fact the space in question is the set of remote points of a certain nowhere locally compact metric space. We show that this result can be considerably generalized by showing that if X is a nowhere locally compact separable metric space, then the set of remote points of βX can be embedded densely in the set of remote points of $\beta\mathbb{Q}$ (where $\mathbb{Q}$ denotes the rationals), and then showing that the set of remote points of $\beta\mathbb{Q}$ has a connected compactification. The set of remote points of βX is denoted by $T(\beta X)$. We summarize some known properties of remote points that we will need in what follows.

Lemma 4.5. (1) *If X is a nowhere locally compact separable metric space, then $T(\beta X)$ is a dense extremally disconnected subspace of βX which contains every other dense extremally disconnected subspace of βX. (This is an immediate consequence of* 2.6 *of* [17]*; the continuum hypothesis which is invoked there is unnecessary since van Douwen* [3] *showed in ZFC that $T(\beta X)$ is a dense subspace of $\beta X \setminus X$, which in turn is dense in βX as X is nowhere locally compact.)*

(2) *If X is normal, then $T(\beta X)$ and $T(\beta EX)$ are homeomorphic.*

(This is 2.6 *of* [8].)

Lemma 4.6. *If X is a nowhere locally compact separable metric space, then $T(\beta X)$ can be embedded as a dense subspace of $T(\beta\mathbb{Q})$.*

Proof. Let S be a countable dense subset of X. Then S is a countable nowhere locally compact metric space and hence is homeomorphic to $\mathbb{Q}$ (see [18]). Let $j : S \to X$ be the embedding map and let $f : \beta S \to \beta X$ be its Stone extension. If $p \in \beta S \setminus T(\beta S)$ then there is a closed nowhere dense subset A of S such that $p \in \mathrm{cl}_{\beta S} A$; consequently, $f(p) \in \mathrm{cl}_{\beta X} f[A] = \mathrm{cl}_{\beta X}(\mathrm{cl}_X j[A])$, and since $\mathrm{cl}_X j[A]$ is a closed nowhere dense subset of X it follows that $f(p) \notin T(\beta X)$. Thus $f^{\leftarrow}[T(\beta X)] \subseteq T(\beta S)$. Now $T(\beta X)$ is dense in βX by 4.5(1) and f is irreducible since j is a dense embedding. Hence it follows from 4.1(4) that $f^{\leftarrow}[T(\beta X)]$ is a dense subspace of $T(\beta S)$ and that $f|f^{\leftarrow}[T(\beta X)]$ is a perfect irreducible map from $f^{\leftarrow}[T(\beta X)]$ onto $T(\beta X)$. Since perfect irreducible surjections onto extremally disconnected spaces are homeomorphisms (see 6.5(d) of [17]), by 4.5(1) and the preceding $T(\beta X)$ is densely embedded in $T(\beta S)$. □

Theorem 4.7. *If X is a nowhere locally compact separable metric space then $T(\beta X)$ is an extremally disconnected space with a connected compactification.*

Proof. Let M be the space $[0,1]^2 \setminus \mathbb{J}^2$, where $\mathbb{J}$ denotes the space of irrationals in $[0,1]$. Clearly $M = (\mathbb{Q} \times [0,1]) \cup ([0,1] \times \mathbb{Q})$ and hence is σ-compact. As $\mathbb{J}^2$ is dense in $[0,1]^2$, M is nowhere locally compact. As any two points of M belong to two homeomorphs of $[0,1]$ whose intersection is nonempty, clearly M is connected. Obviously, $\mathbb{Q}$ is σ-compact and nowhere locally compact also. But 3.1 of [15] asserts that any σ-compact nowhere locally compact metric space is a perfect, irreducible continuous image of $\mathbb{Q} \times \mathbb{C}$, where $\mathbb{C}$ is the Cantor set. It follows from 4.1(2) that $\mathbb{Q}$ and M are coabsolute. By 4.5(2) this in turn implies that $T(\beta\mathbb{Q})$ and $T(\beta M)$ are homeomorphic. But by 4.5(1) $T(\beta M)$ is dense in βM, which is connected because M is. Thus $T(\beta\mathbb{Q})$ has a connected compactification, and by 4.6 $T(\beta X)$ has one as well. □

Observe that as $\beta\mathbb{Q}$ has countable π-weight, $T(\beta\mathbb{Q})$ has a countable dense subset L. Clearly L has a connected compactification and its points are G_δ-sets, and by 4.1(5) L is extremally disconnected. Hence "first countable" cannot be replaced by "singletons are G_δ-sets" in 4.4.

5. Nondense embeddings

Watson and Wilson [20, Problems 5.2 and 5.4] ask which spaces can be embedded as closed (open) subspaces of connected separable spaces.

Lemma 5.1. *Let X be a space (respectively Tychonoff space). Then X can be embedded as a subspace of a connected separable space (respectively Tychonoff space) iff X can be embedded as a closed subspace of a connected separable space (respectively Tychonoff space).*

Proof. The proof in one direction is trivial. Conversely, let $X \subseteq Z$, where Z is a connected separable space and $Y = ([0,1] \times Z) \setminus (\{1\} \times (Z \setminus X))$. The space Y is connected and separable since the subspace $[0,1) \times Z$ is connected and separable, and X is homeomorphic to the closed subspace $\{1\} \times X$ of Y. Now, Y is Hausdorff since Z and $[0,1]$ are; if Z were Tychonoff, Y would be also. □

Theorem 5.2. *The following are equivalent for a space X.*

(1) *X can be embedded as a closed subspace of a connected separable space.*
(2) *X can be embedded as a subspace of a connected separable space.*
(3) *X can be embedded in a separable space.*

Proof. Clearly (1) $\Rightarrow$ (2) $\Rightarrow$ (3). To prove (3) $\Rightarrow$ (1), let $X \subseteq T$, where T is a separable space. Let Y be the quotient of $T \times [0,1]$ formed by collapsing $T \times \{1\}$ to a point p; i.e., the cone over T. The cone Y is always connected, and Y is readily seen to be Hausdorff and separable if T is.

Now, $Y \setminus [(T \setminus X) \times \{0\}]$ is a connected separable space containing X as a closed subspace. □

For a space X, a subset $\mathcal{A} \subseteq \tau(X)$ is a *separating family* if each pair of distinct points of X is contained in disjoint sets of $\mathcal{A}$, and the *separating weight* of X, denoted as $sw(X)$, is the least cardinal of a separating family.

Lemma 5.3. *If the space (X,τ) can be embedded in a separable space Y, then X has a coarser Hausdorff topology τ' with $w(X,\tau') \leqslant \mathfrak{c}$, in particular, $sw(X,\tau) \leqslant \mathfrak{c}$.*

Proof. The semiregularization $Y(s)$ of Y (obtained by retopologizing Y by using the set $\mathcal{RO}(Y)$ of regular open sets of Y as a base) is also separable and Hausdorff. Thus, $wY(s) \leqslant \mathfrak{c}$ as $V \mapsto V \cap S$ is a bijection from $\mathcal{RO}(Y)$ to the power set of the countable dense subset S. Let X' denote X with the subspace topology τ' of $Y(s)$. Now $\tau' \subseteq \tau$, X' is a Hausdorff space and $wX' \leqslant \mathfrak{c}$. □

Lemma 5.4. *If Z is a space with $wZ \leqslant \mathfrak{c}$, there is an extension eS of a countable space S such that Z is homeomorphic to $eS \setminus S$.*

Proof. Let $F = \{0, 1, 2\}$ with $\tau(F) = \{\emptyset, \{0\}, \{2\}, F\}$; $\tau(F)$ is not a Hausdorff topology. Let $R = \{0, 1\}$ and $D = \{0, 2\}$ be subspaces of F. $D^{\mathfrak{c}}$ is dense in $F^{\mathfrak{c}}$; for $a \in F$, let $\underline{a} \in F^{\mathfrak{c}}$ be defined by $\underline{a}(\alpha) = a$ for all $\alpha < \mathfrak{c}$. Also, $D^{\mathfrak{c}}$ has a countable dense subspace T such that $T \subseteq D^{\mathfrak{c}} \setminus \{\underline{0}, \underline{2}\}$. By 2.3.26 in [6], we can assume that Z is a subspace of $R^{\mathfrak{c}} \setminus \{\underline{0}\}$. Note that Z is homeomorphic to $Z \times \{\underline{2}\} \subseteq F^{\mathfrak{c}} \times F^{\mathfrak{c}}$. Also $T \times T$ is dense in $F^{\mathfrak{c}} \times F^{\mathfrak{c}}$ and $(Z \times \{\underline{2}\}) \cup (T \times T)$ is a Hausdorff space; it can serve as the required S. □

Definition 5.5. Let Y be an extension of a space X. Let σ be the topology on Y generated by

$$\tau(Y) \cup \tau(X) \bigcup \{\{p\} \cup U\colon\ p \in Y \setminus X,\ U \in \tau(X),\ \text{and}\ p \in Y \setminus \mathrm{cl}_Y(X \setminus U)\}.$$

(Y, σ) is denoted by Y^+, $\tau(Y^+) \supseteq \tau(Y)$, and Y^+ is an extension of X (see [17, Chapter 7]).

Lemma 5.6. *Let W be an extension of a space X and Z be $W \setminus X$ with a finer topology. Then there is an extension W' of X such that $W' \setminus X = Z$.*

Proof. Define a finer topology σ on W by $U \in \sigma$ if $U \in \tau(W^+)$ and $U \setminus X \in \tau(Z)$. Let W' denote W with σ. W' has the required properties. □

Theorem 5.7. *A space X can be embedded in a separable connected space if and only if $sw(X) \leqslant \mathfrak{c}$.*

Proof. The proof follows from 5.3, 5.4, and 5.6. □

Theorem 5.8. *Let X be Tychonoff, λ a cardinal, and $\mu = 2^\lambda$. The following are equivalent:*

(1) *X can be embedded as closed subspace of a connected Tychonoff space with density character at most λ.*

(2) *X can be embedded in a Tychonoff space with density character at most λ.*

(3) *$w(X) \leqslant \mu$.*

Proof. (1) $\Rightarrow$ (2) is trivial.

(2) $\Rightarrow$ (3): Suppose X is a subspace of a Tychonoff space Y with $dY \leqslant \lambda$. Then $w(Y) \leqslant 2^\lambda = \mu$ (see 2N(2)(d) in [17]). Therefore $w(X) \leqslant w(Y) \leqslant \mu$.

(3) $\Rightarrow$ (1): If $wX \leqslant \mu$, then X can be embedded in $[0, 1]^\mu$. The subspace $Y = (X \times \{0\}) \cup ([0, 1]^\mu \times (0, 1])$ of $[0, 1]^{\mu+1}$ is Tychonoff and connected. The density character of Y is at most λ, and Y contains X as a closed subspace. □

Theorem 5.9. *A space X can be embedded as an open subspace of a connected separable space iff X is separable and has no proper nonempty open H-closed subspaces.*

Proof. Suppose X can be embedded as an open subspace of a connected separable space. Open subspaces of separable spaces are separable, so X must be separable. Now suppose that X has an open proper nonempty H-closed subspace H. If X is embedded in T as an open subspace, then H is open in T as it is open in X. As H is H-closed, it is closed in T. Therefore, H is clopen in T, and as $\emptyset \neq H \neq T$, T is not connected. Hence, if X *can* be embedded as an open subspace of a connected space T, no such H can exist. Conversely, suppose X is separable and has no proper nonempty open H-closed subspaces. Let Y be the cone over κX (see the proof of 5.2). Now Y is separable, connected, and Hausdorff. Also, the subspace $\kappa X \times (0,1]$ of Y is separable and connected. Let σ be the topology on Y generated by $\tau(Y) \cup \{X \times \{0\}\}$. It readily follows that (Y,σ) is separable Hausdorff and contains X as an open subspace. As X is open in κX, the topology on $\kappa X \times \{0\}$ is the same relative to either the topology induced by $\tau(Y)$ or σ. Let C be a clopen subset of (Y,σ). As $\kappa X \times (0,1] \cup (\kappa X \setminus X) \times \{0\}$ is a connected subspace of (Y,σ), either C or $Y \setminus C$ is a clopen subset $\kappa X \times \{0\}$ (and hence is H-closed) and contained in $X \times \{0\}$. By hypothesis, either C or $Y \setminus C$ is empty. Thus, (Y,σ) is connected. □

References

[1] M.G. Bell, On the combinatorial principle P(𝔠), Fund. Math. 114 (1981) 149–157.
[2] W.W. Comfort and S. Negrepontis, The Theory of Ultrafilters (Springer, New York, 1974).
[3] E. van Douwen, Remote Points, Dissertationes Math. 188 (1981) 1–45.
[4] E. van Douwen, The integers and topology, in: Handbook of Set-Theoretic Topology (North-Holland, Amsterdam, 1984) 111–167.
[5] A. Emeryk and W. Kulpa, The Sorgenfrey line has no connected compactification, Comm. Math. Univ. Carolin. 18 (1977) 483–487.
[6] R. Engelking, General Topology, Sigma Series in Pure Mathematics 6 (Heldermann Verlag, Berlin, 1989).
[7] Z. Frolík, A generalization of realcompact spaces, Czechoslovak Math. J. 13 (1963) 127–137.
[8] C.L. Gates, Some structural properties of the set of remote points of a metric space, Canad. J. Math. 32 (1980) 195–209.
[9] L. Gillman and M. Jerison, Rings of Continuous Functions (Van Nostrand Reinhold, New York, 1960).
[10] K.P. Hart, The Čech–Stone compactification of the real line, in: Recent Progress in General Topology (North-Holland, Amsterdam, 1992) 317–352.
[11] J.R. Isbell, Uniform Spaces, Mathematical Surveys 12 (American Mathematical Society, Providence, RI, 1964).
[12] J. Mack, The long line as a remainder, in: Rings of Continuous Functions, Lecture in Notes Pure and Applied Mathematics 95 (Marcel Dekker, New York, 1985) 237–245.
[13] K.D. Magill Jr, A note on compactifications, Math. Z. 94 (1966) 322–325.
[14] J. van Mill and R.G. Woods, Perfect images of zero-dimensional separable metric spaces, Canad. Math. Bull. 25 (1982) 41–47.
[15] D. Plank, On a class of subalgebras of $C(X)$ with applications to $\beta X \setminus X$, Fund. Math. 64 (1969) 41–45.

[16] J.R. Porter and J. Vermeer, Spaces with coarser minimal Hausdorff topologies, Trans. Amer. Math. Soc. 289 (1985) 59–71.
[17] J.R. Porter and R.G. Woods, Extensions and absolutes of Hausdorff spaces (Springer, New York, 1987).
[18] S. Sierpinski, Sur une proprieté topologique des ensembles denumerable en soi, Fund. Math. 1 (1920) 44–60.
[19] A. Tychonoff, Über die topologische Erweiterung von Räumen, Math. Ann. 102 (1930) 544–561.
[20] S.W. Watson and R.G. Wilson, Embeddings in connected spaces, Houston J. Math. 19 (1993) 469–481.
[21] R.G. Woods, Ideals of pseudocompact regular closed sets and absolutes of Hewitt realcompactifications, Gen. Topology Appl. 2 (1972) 315–331.